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Tunnelling of scalar and Dirac particles from squashed charged rotating Kaluza–Klein black holes

M. M. Stetsko[a]

Department of Theoretical Physics, Ivan Franko National University of Lviv, 12 Drahomanov Str., Lviv 79005, Ukraine

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Abstract The thermal radiation of scalar particles and Dirac fermions from squashed charged rotating five-dimensional black holes is considered. To obtain the temperature of the black holes we use the tunnelling method. In the case of scalar particles we make use of the Hamilton–Jacobi equation. To consider tunnelling of fermions the Dirac equation was investigated. The examination shows that the radial parts of the action for scalar particles and fermions in the quasi-classical limit in the vicinity of horizon are almost the same and as a consequence it gives rise to identical expressions for the temperature in the two cases.

1 Introduction

Hawking radiation has been investigated since the early 1970s [1,2]. To find the temperature of black holes different methods were used. The semiclassical tunnelling method proposed by Kraus and Wilczek [3,4] and developed in work of Parikh and Wilczek [5] has attracted a lot of interest recently. We remark that nowadays the tunnelling method comprises two different approaches, namely the approach proposed by Parikh and Wilczek, called the null-geodesic method, and one that is known as the Hamilton–Jacobi method [6,7]. It is worth noting that the null-geodesic method is truly semiclassical because it is based on the equation of motion of a classical particle that moves along a null-geodesic. The Hamilton–Jacobi method is based on equations that we deal with in quantum theory such as the Klein–Gordon equation for scalar particles and the Dirac equation for fermions with spin $s = 1/2$. For particles of higher spins the Rarita–Schwinger or Proca equations can be used. So here one departs from a quantum equation of motion and then considers the quasi-classical limit. Despite the different starting points two methods give rise to the same temperatures if the black hole's metrics are the same.

These approaches were applied to numerous examples of black holes and the obtained results were in agreement with the expressions for the temperature calculated by other methods. Among them we distinguish Kerr and Kerr–Newman black holes' space-times [8–12], Taub-NUT space-time [13], Gödel space-time [14], BTZ black holes [15,16], dynamical black holes [17], black holes in Hořava–Lifshitz gravity [18,19], accelerating and rotating black holes [20,21], rotating black strings [22], and many others. A review of the tunnelling method is presented in Ref. [23], where further references on that subject are given.

Multidimensional black holes have attracted great attention in recent years [24]. There are great expectations that multidimensional micro black holes can appear in high energy particles collisions, which are based on the fact that extra spacelike dimensions can lower the Planck scale up to the TeV-energy region [25–29]. Squashed Kaluza–Klein black holes represent one of the most interesting solutions among higher-dimensional black holes. The first five-dimensional Kaluza–Klein black hole's solution was found by Dobiasch and Maison [30] and the metrics involved were studied in [31,32]. The solution was generalised to the five-dimensional Einstein–Maxwell theory [33]. To find the so-called squashing solution a transformation was used [33]. The squashing transformation was applied to construct a rotating black hole solution [34], a charged rotating black hole [35], and a charged rotating black hole in a Gödel universe [36,37]. A review of Kaluza–Klein black holes' solutions can be found in [38].

Different aspects of Kaluza–Klein black holes were also considered, namely thermodynamics [39–41], Hawking radiation and the tunnelling method [42–48], quasinormal modes and stabilities [49–53], geodetic precession [54] and gravitational lensing [55].

In our work we consider scalar particles and fermion tunnelling for charged rotating Kaluza–Klein black holes. The paper is organised as follows. In the second section we inves-

[a] e-mail: mstetsko@gmail.com; mykola@ktf.franko.lviv.ua

tigate the tunnelling of scalar particles. In the third section the tunnelling of Dirac fermions is examined. The fourth section contains some conclusions.

2 Tunnelling of scalar particles

Let us consider the tunnelling of scalar particles. The Klein–Gordon equation for charged massive particles can be written in the form

$$\frac{1}{\sqrt{-g}}\left(\partial_\mu - ieA_\mu\right)\sqrt{-g}g^{\mu\nu}\left(\partial_\nu - ieA_\nu\right)\Psi - \tilde{m}^2\Psi = 0. \tag{1}$$

The tunnelling process is supposed to be quasi-classical. To consider it we assume that the quasi-classical wave function takes the form

$$\Psi = C\exp\left\{\frac{i}{\hbar}I_\uparrow\right\}. \tag{2}$$

Here $I_\uparrow = I_\uparrow(t, r, \theta, \phi, \psi)$ denotes a quasi-classical action of the emitted particles. In the first order approximation the Klein–Gordon Eq. (1) leads to a Hamilton–Jacobi equation for the relativistic particle:

$$g^{\mu\nu}\left(\partial_\mu I_\uparrow \partial_\nu I_\uparrow + e^2 A_\mu A_\nu - 2eA_\mu \partial_\nu I_\uparrow\right) + \tilde{m}^2 = 0. \tag{3}$$

We will examine tunnelling of particles through the horizon of a squashed charged rotating Kaluza–Klein black hole whose metric and gauge potential were obtained in [35]:

$$\begin{aligned} ds^2 = &-\frac{w(r)}{h(r)}dt^2 + \frac{k^2(r)}{w(r)}dr^2 \\ &+ \frac{r^2}{4}\left[k(r)(\sigma_1^2 + \sigma_2^2) + h(r)(f(r)dt + \sigma_3)^2\right] \end{aligned} \tag{4}$$

where the functions $w(r)$, $h(r)$, $k(r)$ and $f(r)$ are defined as follows:

$$\begin{aligned} w(r) &= \frac{(r^2+q)^2 - 2(m+q)(r^2-a^2)}{r^4} \\ &= \frac{(r^2-r_+^2)(r^2-r_-^2)}{r^4}, \end{aligned} \tag{5}$$

$$h(r) = 1 - \frac{a^2q^2}{r^6} + \frac{2a^2(m+q)}{r^4}, \tag{6}$$

$$\begin{aligned} k(r) &= \frac{(r_\infty^2+q)^2 - 2(m+q)(r_\infty^2-a^2)}{(r_\infty^2-r^2)^2} \\ &= \frac{(r_\infty^2-r_+^2)(r_\infty^2-r_-^2)}{(r_\infty^2-r^2)^2}, \end{aligned} \tag{7}$$

$$f(r) = -\frac{2a}{r^2h(r)}\left(\frac{2m+q}{r^2} - \frac{q^2}{r^4}\right). \tag{8}$$

The gauge potential is defined by the relation

$$A = \frac{\sqrt{3}q}{2r^2}\left(dt - \frac{a}{2}\sigma_3\right) \tag{9}$$

and left-invariant 1-forms on S^3 are given by

$$\sigma_1 = \cos\psi d\theta + \sin\psi\sin\theta d\phi, \tag{10}$$

$$\sigma_2 = -\sin\psi d\theta + \cos\psi\sin\theta d\phi, \tag{11}$$

$$\sigma_3 = d\psi + \cos\theta d\phi. \tag{12}$$

The coordinates $(t, r, \theta, \phi, \psi)$ run over the ranges of $-\infty < t < +\infty$, $0 < r < r_\infty$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$, $0 \le \psi < 4\pi$, respectively. The case $k(r) = 1$ or equivalently $r_\infty \to \infty$ leads to the Cvetič solution [56,57]. The parameters a, m, q and r_∞ fulfill the inequalities

$$m > 0, \tag{13}$$

$$q^2 + 2(m+q)a^2 > 0, \tag{14}$$

$$(r_\infty^2+q)^2 - 2(m+q)(r_\infty^2-a^2) > 0, \tag{15}$$

$$(m+q)(m-q-2a^2) > 0, \tag{16}$$

$$m+q > 0. \tag{17}$$

It was pointed out by the authors of [35] that the inequalities (13)–(16) are necessary for the existence of two horizons and the last inequality (17) provides the absence of closed timelike curves outside the outer horizon. The horizons are defined by the equation $w(r) = 0$ and it leads to $r_+^2 = m + \sqrt{(m+q)(m-q-2a^2)}$ and $r_-^2 = m - \sqrt{(m+q)(m-q-2a^2)}$. The metric (4) also diverges at $r = r_\infty$ but it is an apparent singularity corresponding to the spatial infinity. To prove it one should make a coordinate transformation,

$$\rho = \rho_0\frac{r^2}{r_\infty^2 - r^2}, \tag{18}$$

where ρ_0 is defined by

$$\begin{aligned} \rho_0^2 &= \frac{(r_\infty^2+q)^2 - 2(m+q)(r_\infty^2-a^2)}{4r_\infty^2} \\ &= \frac{(r_\infty^2-r_+^2)(r_\infty^2-r_-^2)}{4r_\infty^2}. \end{aligned} \tag{19}$$

It is clear when $r \to r_\infty$ then $\rho \to \infty$. It was also shown [35] that the asymptotic time (time for a distant observer) differs from the coordinate one and takes the form

$$\tilde{t} = \frac{2r_\infty^2\rho_0}{\sqrt{r_\infty^6 - a^2(q^2 - 2(m+q)r_\infty^2)}}t = \beta t. \tag{20}$$

Now we consider the Hamilton–Jacobi Eq. (3) in case of the black hole's metric (4). The functions $w(r)$, $k(r)$, $h(r)$ and $f(r)$ depend only on the radial variable r and this gives rise to the conclusion that the angular variables can be separated form the radial one (or at least some of them). Using this fact we suppose that the angular variables ϕ and ψ can be separated. So the action for the emitted particle $I_\uparrow$ might be written in the form

$$I_\uparrow = -E\tilde{t} + W(r, \theta) + J\phi + L\psi \tag{21}$$

where J, E and L are constants and $\tilde{t}$ denotes the time for a distant observer which is defined by the relation (20).

As a result the Hamilton–Jacobi equation can be represented in the form

$$\frac{w(r)}{k^2(r)}W_r^2+\frac{4W_\theta^2}{r^2k(r)} -\frac{h(r)}{w(r)}\left(\beta E+f(r)L+\frac{\sqrt{3}qe}{2r^2}\left(1+\frac{af(r)}{2}\right)\right)^2 +\frac{1}{r^2h(r)}\left(2L+\frac{\sqrt{3}qea}{2r^2}\right)^2+\frac{4}{r^2k(r)}\left(\frac{J}{\sin\theta}-L\cot\theta\right)^2 +\tilde{m}^2=0. \tag{22}$$

One can see that the variables r and θ can be separated in the same way. So the function $W(r,\theta)$ can be written as follows:

$$W(r,\theta)=R(r)+\Theta(\theta). \tag{23}$$

Then for the radial part we obtain

$$R'(r) =\frac{\beta k(r)\sqrt{h(r)}}{w(r)}\left[\left(E+\frac{f(r)}{\beta}L+\frac{\sqrt{3}qe}{2\beta r^2}\left(1+\frac{af(r)}{2}\right)\right)^2 -\frac{w(r)}{\beta^2h(r)}\left(\frac{1}{r^2h(r)}\left(2L+\frac{\sqrt{3}qea}{2r^2}\right)^2+\frac{4D}{r^2k(r)}+\tilde{m}^2\right)\right]^{1/2} \tag{24}$$

where D is a constant that appeared after separation of the variables.

In the vicinity of the horizon point $w(r_+)=0$ one should use the decomposition $w(r)=w'(r_+)(r-r_+)=2(r_+^2-r_-^2)(r-r_+)/r_+^3$. Then the latter equation can be written in the form

$$R'(r)=\frac{\beta}{2}\frac{r_\infty^2-r_-^2}{r_\infty^2-r_+^2}\frac{\sqrt{r_+^6-a^2q^2-2a^2(m+q)r_+^2}}{(r_+^2-r_-^2)(r-r_+)} \times\left(E+\frac{f(r_+)L}{\beta}+\frac{\sqrt{3}qe}{2\beta r_+^2}\left(1+\frac{af(r_+)}{2}\right)\right). \tag{25}$$

Integrating this expression in the vicinity of the horizon point we obtain

$$R(r)=\frac{\beta}{2}\frac{r_\infty^2-r_-^2}{r_\infty^2-r_+^2}\frac{\sqrt{r_+^6-a^2q^2-2a^2(m+q)r_+^2}}{(r_+^2-r_-^2)} \times\left(E+\frac{f(r_+)L}{\beta}+\frac{\sqrt{3}qe}{2\beta r_+^2}\left(1+\frac{af(r_+)}{2}\right)\right) \times\int_{r_+-\varepsilon}^{r_++\varepsilon}\frac{\mathrm{d}r}{(r-r_+)}. \tag{26}$$

We consider the tunnelling process and it means that the action for the emitted particles $I_\uparrow$ (21) takes complex values. This fact immediately follows from the form of the integral for the radial part of the action (26). The function we integrate in (26) has a pole at the point r_+. Then we should integrate around the pole and as a result the complex values appear. The imaginary part of the radial part of the action (26) is of the crucial importance when one tries to obtain the temperature of the black hole. So we can write

$$\mathrm{Im}R_\uparrow=\pi\frac{\beta}{2}\frac{r_\infty^2-r_-^2}{r_\infty^2-r_+^2}\frac{\sqrt{r_+^6-a^2q^2-2a^2(m+q)r_+^2}}{(r_+^2-r_-^2)} \times\left(E+\frac{f(r_+)L}{\beta}+\frac{\sqrt{3}qe}{2\beta r_+^2}\left(1+\frac{af(r_+)}{2}\right)\right). \tag{27}$$

It was supposed [6,23] that the probabilities of crossing a black hole's horizon can be defined by

$$P_{\mathrm{out}}\propto\exp\{-2\mathrm{Im}R_\uparrow\},\quad P_{\mathrm{int}}\propto\exp\{-2\mathrm{Im}R_\downarrow\}. \tag{28}$$

The ratio for these two probabilities leads to the Boltzmann factor $\exp\{-E/T\}$, which shows that the radiation is thermal [58]:

$$\Gamma=\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}=\frac{\exp\{-2\mathrm{Im}R_\uparrow\}}{\exp\{-2\mathrm{Im}R_\downarrow\}}=\exp\left\{-\frac{E}{T}\right\}. \tag{29}$$

The imaginary part of the radial part of the action for ingoing particles can be calculated in the same way as it was carried out for the outgoing case. The resulting imaginary part will take only the opposite sign in comparison to Eq. (27), so $\mathrm{Im}R_\downarrow=-\mathrm{Im}R_\uparrow$. Substituting Eq. (26) and taking into consideration the remark mentioned above we arrive at

$$\Gamma=\exp\left\{-2\pi\beta\frac{r_\infty^2-r_-^2}{r_\infty^2-r_+^2}\frac{\sqrt{r_+^6-a^2q^2-2a^2(m+q)r_+^2}}{(r_+^2-r_-^2)} \times\left(E+\frac{f(r_+)L}{\beta}+\frac{\sqrt{3}qe}{2\beta r_+^2}\left(1+\frac{af(r_+)}{2}\right)\right)\right\}. \tag{30}$$

Having supposed that the emission of particles is thermal we obtain the temperature of the black hole:

$$T=\frac{1}{2\pi}\frac{r_+^2-r_-^2}{r_\infty(r_\infty^2-r_-^2)}\sqrt{\frac{r_\infty^2-r_+^2}{r_\infty^2-r_-^2}} \times\sqrt{\frac{r_\infty^6-a^2(q^2-2(m+q)r_\infty^2)}{r_+^6-a^2(q^2-2(m+q)r_+^2)}}. \tag{31}$$

The result we obtain is in agreement with the expression for the surface gravity that was found in [35]. It was argued that imaginary contribution to the action integral can be caused also by the temporal part [59,60]. Taking into account the imaginary contribution from the temporal part prevents the situation when the probability for ingoing particles might exceed the unit. But the tunnelling probability and as a consequence the temperature of the black hole take the same form as we have obtained.

Now we consider a squashed black hole in the Gödel universe. The metric of the black hole takes the following form [36]:

$$ds^2 = -k(r)dt^2 - 2g(r)\sigma_3 dt + h(r)\sigma_3^2 + \frac{\chi^2(r)}{V(r)}dr^2 + \frac{r^2}{4}\left[\chi(r)(\sigma_1^2+\sigma_2^2)\right] \tag{32}$$

where

$$k(r) = 1 - \frac{2m}{r^2} + \frac{q^2}{r^4}, \tag{33}$$

$$g(r) = jr^2 + 3jq + \frac{(2m-q)a}{2r^2} - \frac{q^2 a}{2r^4}, \tag{34}$$

$$h(r) = \frac{r^2}{4} - j^2 r^2 (r^2 + 2m + 6q) + 3jqa + \frac{(m-q)a^2}{2r^2} - \frac{q^2 a^2}{4r^4}; \tag{35}$$

$$V(r) = 1 + \frac{8j(m+q)(a+2j(m+2q)) - 2m}{r^2} + \frac{2(m-q)a^2 + q^2[1 - 16ja - 8j^2(m+3q)]}{r^4}; \tag{36}$$

$$\chi(r) = \frac{c^2+2c(m-4j(m+q)[a+2j(m+2q)])+q^2+2a^2(m-q)-8q^2 j[2a+j(m+3q)]}{(r^2+c)^2}. \tag{37}$$

We note that here we keep the notation given in Ref. [36]. The squashing function here is $\chi(r)$. It was shown that in the case when the Gödel parameter $j=0$ we arrive at the previously considered metric (4). We also note that the squashing parameter c has to be chosen negative $c=-r_0^2$ (the parameter r_0 corresponds to the parameter r_∞ of the metric previously considered). The electromagnetic one-form potential can be written as follows:

$$A = \left(\frac{\sqrt{3}q}{2r^2} - \Phi\right)dt + \frac{\sqrt{3}}{2}\left(jr^2 + 2jq - \frac{aq}{2r^2}\right)\sigma_3 \tag{38}$$

and here Φ is a constant which one can find from the requirement that electromagnetic potential should be regular at the horizon. Horizon points for the black hole given by the metric (32) can be found as the roots of the equation $V(r)=0$ [36]:

$$r_\pm^2 = m - 4j(m+q)(a+2j(m+2q)) \pm \sqrt{\delta}, \tag{39}$$

$$\delta = (m - q - 8j^2(m+q)^2) \times [m + q - 2a^2 - 8ja(m+2q) - 8j^2(m+2q)^2]. \tag{40}$$

We note that in the limit $j=0$ we recover the previously considered black hole's solution (4).

It should be noticed that black hole solutions in the Gödel universe possess closed timelike curves (CTC). In our case we will consider the situation when $r_- < r_+ < r_0 < r_{CTC}$ so this peculiarity of the solution (32) is not important for us.

Now we examine the Hamilton–Jacobi equation for the squashed Gödel black hole. Similarly to the previous case the components of metric tensor do not depend explicitly on the coordinates t, ϕ, ψ, and this allows us to separate the angular and time coordinates. So the action for an emitted particle can be chosen in the form (21). Having accomplished the separation of the variables we write the Hamilton–Jacobi equation:

$$\frac{V(r)}{\chi^2(r)}W_r^2 + \frac{4}{r^2\chi(r)}\left(W_\theta^2 + \frac{(J - L\cos\theta)^2}{\sin^2\theta}\right) + \frac{4}{r^2 V(r)}\left(-h(r)\left[E + e\left(\frac{\sqrt{3}q}{2r^2} - \Phi\right)\right]^2 + 2g(r)\left[\left(E + e\left(\frac{\sqrt{3}q}{2r^2} - \Phi\right)\right) \times \left(L - \frac{\sqrt{3}e}{2}\left(jr^2 + 2jq - \frac{aq}{2r^2}\right)\right)\right] + k(r)\left(L - \frac{\sqrt{3}e}{2}\left(jr^2+2jq-\frac{aq}{2r^2}\right)\right)^2\right) + \tilde{m}^2 = 0. \tag{41}$$

We note that here we have used the relation $g^2(r) + k(r)h(r) = r^2 V(r)/4$, which can be verified easily. Similarly to the previous case the radial r and angular θ variables can be separated. So we make use of the relation (23) and write the relation for the derivative of the radial part of the action $I_\uparrow$,

$$R'(r) = \frac{2\chi(r)\sqrt{h(r)}}{rV(r)} \left(\left[E + e\left(\frac{\sqrt{3}q}{2r^2} - \Phi\right) - \frac{g(r)}{h(r)}\left(L - \frac{\sqrt{3}e}{2}\left(jr^2 + 2jq - \frac{aq}{2r^2}\right)\right)\right]^2 - \frac{r^2V(r)}{h(r)}\left[\frac{1}{h(r)}\left(L - \frac{\sqrt{3}e}{2}\left(jr^2 + 2jq - \frac{aq}{2r^2}\right)\right)^2 + \frac{4D}{r^2\chi(r)} + \tilde{m}^2\right]\right)^{1/2}. \quad (42)$$

Integrating the obtained relation around the horizon point r_+ and taking the imaginary part of it we arrive at

$$\mathrm{Im} R_\uparrow = \pi \frac{r_+^2 \chi(r_+)\sqrt{h(r_+)}}{(r_+^2 - r_-^2)} \left[E + e\left(\frac{\sqrt{3}q}{2r_+^2} - \Phi\right) - \frac{g(r_+)}{h(r_+)}\left(L - \frac{\sqrt{3}e}{2}\left(jr_+^2 + 2jq - \frac{aq}{2r_+^2}\right)\right)\right]. \quad (43)$$

To find the temperature of the black hole we make use of the same procedure as we have done earlier. It should be emphasised that similarly to the previous metric to obtain the correct relation for the temperature we have to take into account the fact that a distant observer uses the asymptotic time but not a coordinate one. So we perform a transformation similar to (20) $t \to t/N_0$, where the parameter N_0 takes the form

$$N_0^2 = \frac{r_0^2 V(r_0)}{4h(r_0)}. \quad (44)$$

This transformation leads to the replacement $E \to EN_0$ in the right hand side of Eq. (43). We note that in the case of the previously considered metric the situation was the same.

So we write

$$T = \frac{1}{2\pi}\frac{r_0}{r_+^2}\frac{r_+^2 - r_-^2}{r_0^2 - r_-^2}\sqrt{\frac{r_0^2 - r_+^2}{r_0^2 - r_-^2}}\sqrt{\frac{h(r_0)}{h(r_+)}}. \quad (45)$$

In the limit when the Gödel parameter is equal to zero ($j = 0$) the obtained relation (45) can be represented in the form (31).

3 Tunnelling of a charged spin-1/2 particle from squashed Kaluza–Klein black hole

In this section we examine the tunnelling method for Dirac particles. For the first time the tunnelling method for fermions was considered by Kerner and Mann [11]. Then it was successfully applied to the vast area of black holes [23]. In the case of scalar particles the starting point was the Klein–Gordon Eq. (1), which leads to the Hamilton–Jacobi Eq. (3) in the quasi-classical limit. To investigate the tunnelling of fermions the Klein–Gordon equation should be replaced by the Dirac equation. Then similarly to the case of scalar particles the quasi-classical limit should be taken. First of all we consider the black hole's metric (4) and then go to a bit more general metric of the squashed black hole in a Gödel universe (32). The Dirac equation for an electrically charged particle takes the form

$$i\gamma^\mu\left(D_\mu - \frac{ie}{\hbar}A_\mu\right)\Psi + \frac{\tilde{m}}{\hbar}\Psi = 0 \quad (46)$$

where $D_\mu = \partial_\mu + \Omega_\mu$, $\Omega_\mu = \frac{1}{8}\Gamma_\mu^{\alpha\beta}[\gamma^\beta, \gamma^\alpha]$ and the γ^μ matrices fulfill the commutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\hat{1}. \quad (47)$$

The matrices γ^μ can be defined in different manners and in our work we take them in the following form:

$$\hat{\gamma}^{\tilde{t}} = \beta\sqrt{\frac{h(r)}{w(r)}}\hat{\gamma}^0, \quad \hat{\gamma}^r = \frac{\sqrt{w(r)}}{k(r)}\hat{\gamma}^3, \quad \hat{\gamma}^\theta = \frac{2}{r\sqrt{k(r)}}\hat{\gamma}^1,$$
$$\hat{\gamma}^\phi = \frac{2}{r\sqrt{k(r)}\sin\theta}\hat{\gamma}^2,$$
$$\hat{\gamma}^\psi = -f(r)\sqrt{\frac{h(r)}{w(r)}}\hat{\gamma}^0 - \frac{2\cot\theta}{r\sqrt{k(r)}}\hat{\gamma}^2 + \frac{2}{r\sqrt{h(r)}}\hat{\gamma}^4. \quad (48)$$

It should be noted that we consider the black hole's metric (4). The matrices γ^A take the form

$$\gamma^0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \quad (49)$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \quad \gamma^4 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \quad (50)$$

and the σ_i are the Pauli matrices.

So the Dirac equation can be written in the form

$$i\left[\beta\sqrt{\frac{h(r)}{w(r)}}\hat{\gamma}^0\partial_{\tilde{t}} + \frac{\sqrt{w(r)}}{k(r)}\hat{\gamma}^3\partial_r + \frac{2}{r\sqrt{k(r)}}\hat{\gamma}^1\partial_\theta + \frac{2}{r\sqrt{k(r)}\sin\theta}\hat{\gamma}^2\partial_\phi + \left(-f(r)\sqrt{\frac{h(r)}{w(r)}}\hat{\gamma}^0 - \frac{2\cot\theta}{r\sqrt{k(r)}}\hat{\gamma}^2 + \frac{2}{r\sqrt{h(r)}}\hat{\gamma}^4\right)\partial_\psi\right]\Psi$$

$$+\frac{\sqrt{3}qe}{2\hbar r^2}\left[\sqrt{\frac{h(r)}{w(r)}}\left(1+\frac{af(r)}{2}\right)\hat{\gamma}^0-\frac{a}{r\sqrt{h(r)}}\hat{\gamma}^4\right]\Psi$$
$$+\frac{\tilde{m}}{\hbar}\Psi=0. \tag{51}$$

We also remark that here the spin connection terms Ω_μ are omitted because we will consider the quasi-classical limit and take the lowest order approximation, whereas the spin connection gives rise to the terms of the next order.

The wave functions with spin up and down can be chosen in the form

$$\Psi_\uparrow=\begin{pmatrix}A(t,r,\theta,\phi,\psi)\\0\\B(t,r,\theta,\phi,\psi)\\0\end{pmatrix}\exp\left(\frac{i}{\hbar}I_\uparrow(t,r,\theta,\phi,\psi)\right);$$
$$\Psi_\downarrow=\begin{pmatrix}0\\C(t,r,\theta,\phi,\psi)\\0\\B(t,r,\theta,\phi,\psi)\end{pmatrix}\exp\left(\frac{i}{\hbar}I_\downarrow(t,r,\theta,\phi,\psi)\right), \tag{52}$$

where $I_\uparrow$ and $I_\downarrow$ are the action for the Dirac particles with spin up ($\uparrow$) and spin down ($\downarrow$) tunnelling through the horizon.

Substituting (52) into the Dirac Eq. (51) and taking the lowest order terms (the terms proportional to $\hbar^{-1}$) we arrive at

$$B\left(\sqrt{\frac{h(r)}{w(r)}}\left[-\beta\partial_{\tilde{t}}I_\uparrow+f(r)\partial_\psi I_\uparrow+\frac{\sqrt{3}qe}{2r^2}\left(1+\frac{af(r)}{2}\right)\right]\right.$$
$$\left.-\frac{\sqrt{w(r)}}{k(r)}\partial_r I_\uparrow\right)+A\left(\frac{2}{r\sqrt{h(r)}}\partial_\psi I_\uparrow+\frac{\sqrt{3}qea}{2r^3\sqrt{h(r)}}+\tilde{m}\right)=0, \tag{53}$$

$$\frac{2B}{r\sqrt{k(r)}}\left(-\partial_\theta I_\uparrow-\frac{i}{\sin\theta}\partial_\phi I_\uparrow+i\cot\theta\partial_\psi I_\uparrow\right)=0, \tag{54}$$

$$A\left(\sqrt{\frac{h(r)}{w(r)}}\left[\beta\partial_{\tilde{t}}I_\uparrow-f(r)\partial_\psi I_\uparrow-\frac{\sqrt{3}qe}{2r^2}\left(1+\frac{af(r)}{2}\right)\right]\right.$$
$$\left.-\frac{\sqrt{w(r)}}{k(r)}\partial_r I_\uparrow\right)+B\left(-\frac{2}{r\sqrt{h(r)}}\partial_\psi I_\uparrow-\frac{\sqrt{3}qea}{2r^3\sqrt{h(r)}}+\tilde{m}\right)=0, \tag{55}$$

$$\frac{2A}{r\sqrt{k(r)}}\left(-\partial_\theta I_\uparrow-\frac{i}{\sin\theta}\partial_\phi I_\uparrow+i\cot\theta\partial_\psi I_\uparrow\right)=0. \tag{56}$$

It follows immediately from the obtained equations that all the variables can be separated. The action $I_\uparrow$ is supposed to take the form

$$I_\uparrow=-Et+J\phi+L\psi+R(r)+\Theta(\theta), \tag{57}$$

where E is the energy of the emitted Dirac particles and J and L are the angular momenta corresponding to the angles ϕ and ψ, respectively. We also remark that in the case of scalar particles we initially assumed that just part of the variables can be separated and after a little algebra we concluded that we had a complete separation of variables. The complete separation of the variables for Dirac particles follows from Eqs. (53)–(56). Having inserted the ansatz (57) into Eqs. (53)–(56) we obtain

$$B\left(\sqrt{\frac{h(r)}{w(r)}}\left[\beta E+f(r)L+\frac{\sqrt{3}qe}{2r^2}\left(1+\frac{af(r)}{2}\right)\right]\right.$$
$$\left.-\frac{\sqrt{w(r)}}{k(r)}R'(r)\right)+A\left(\frac{1}{r\sqrt{h(r)}}\left(2L+\frac{\sqrt{3}qea}{2r^2}\right)+\tilde{m}\right)$$
$$=0, \tag{58}$$

$$-\frac{2B}{r\sqrt{k(r)}}\left(\Theta'+\frac{iJ}{\sin\theta}-iL\cot\theta\right)=0, \tag{59}$$

$$A\left(-\sqrt{\frac{h(r)}{w(r)}}\left[\beta E+f(r)L+\frac{\sqrt{3}qQe}{2r^2}\left(1+\frac{af(r)}{2}\right)\right]\right.$$
$$\left.-\frac{\sqrt{w(r)}}{k(r)}R'(r)\right)+A\left(\frac{-1}{r\sqrt{h(r)}}\left(2L+\frac{\sqrt{3}qea}{2r^2}\right)+\tilde{m}\right)$$
$$=0, \tag{60}$$

$$-\frac{2A}{r\sqrt{k(r)}}\left(\Theta'+\frac{iJ}{\sin\theta}-iL\cot\theta\right)=0. \tag{61}$$

The system of Eqs. (58) and (60) has a nontrivial solution for the parameters A and B if and only if the determinant of corresponding matrix is equal to zero. That requirement gives rise to the following equation:

$$R'(r)=\frac{\beta k(r)\sqrt{h(r)}}{w(r)}\left[\left(E+\frac{f(r)}{\beta}L+\frac{\sqrt{3}qe}{2\beta r^2}\left(1+\frac{af(r)}{2}\right)\right)^2\right.$$
$$\left.-\frac{w(r)}{\beta^2h(r)}\left(\frac{1}{r^2h(r)}\left(2L+\frac{\sqrt{3}qea}{2r^2}\right)^2-\tilde{m}^2\right)\right]^{1/2}. \tag{62}$$

The structure of the obtained relation is similar to the relation for scalar particles, (24), and this is not an accidental coincidence because we consider a quasi-classical approximation for both types of particles. In the vicinity of the horizon point we again use the decomposition $w(r)=w'(r_+)(r-r_+)=2(r_+^2-r_-^2)(r-r_+)/r_+^3$. So we can write

$$R'(r)=\frac{\beta}{2}\frac{r_\infty^2-r_-^2}{r_\infty^2-r_+^2}\frac{\sqrt{r_+^6-a^2q^2-2a^2(m+q)r_+^2}}{(r_+^2-r_-^2)(r-r_+)}$$
$$\times\left(E+\frac{f(r_+)L}{\beta}+\frac{\sqrt{3}qe}{2\beta r_+^2}\left(1+\frac{af(r_+)}{2}\right)\right). \tag{63}$$

To obtain the temperature for Dirac particles we follow the same steps as we have made for the scalar particles. The similarity of the obtained relations for the derivatives of the radial part of the action brings us to the conclusion that the temperature for tunnelling fermions has to be the same as for scalar particles and will be defined by Eq. (31). We also note that the equality of the temperatures for scalar particles and Dirac fermions follows from the fact that both types of particles are examined quasi-classically.

Now we proceed to the tunnelling of Dirac particles in the case of the black hole's metric (32). We note that fermion tunnelling from the squashed black hole in the Gödel universe was investigated in [47], but the author considered just the particular case of a nonrotating black hole without charge. To write the Dirac equation gamma matrices should be defined. The gamma matrices take the form

$$\hat{\gamma}^t = \frac{2}{r}\sqrt{\frac{h(r)}{V(r)}}\hat{\gamma}^0, \quad \hat{\gamma}^r = \frac{\sqrt{V(r)}}{\chi(r)}\hat{\gamma}^3, \quad \hat{\gamma}^\theta = \frac{2}{r\sqrt{\chi(r)}}\hat{\gamma}^1,$$
$$\hat{\gamma}^\phi = \frac{2}{r\sqrt{\chi(r)}\sin\theta}\hat{\gamma}^2,$$
$$\hat{\gamma}^\psi = \frac{2g(r)}{r\sqrt{h(r)V(r)}}\hat{\gamma}^0 - \frac{2\cot\theta}{r\sqrt{k(r)}}\hat{\gamma}^2 + \frac{1}{r\sqrt{h(r)}}\hat{\gamma}^4. \tag{64}$$

The Dirac equation can be written as follows:

$$i\left(\frac{2\sqrt{h(r)}}{r\sqrt{V(r)}}\hat{\gamma}^0\partial_t + \frac{\sqrt{V(r)}}{\chi(r)}\hat{\gamma}^3\partial_r + \frac{2}{r\sqrt{\chi(r)}}\hat{\gamma}^1\partial_\theta + \frac{2}{r\sqrt{\chi(r)}\sin\theta}\hat{\gamma}^2\partial_\varphi + \left[\frac{2g(r)}{r\sqrt{h(r)V(r)}}\hat{\gamma}^0 - \frac{2\cot\theta}{r\sqrt{k(r)}}\hat{\gamma}^2 + \frac{1}{\sqrt{h(r)}}\hat{\gamma}^4\right]\partial_\psi\right)$$
$$\Psi + \frac{e}{\hbar}\left(\frac{2\sqrt{h(r)}}{r\sqrt{V(r)}}\hat{\gamma}^0 \times \left(\frac{\sqrt{3}q}{2r^2} - \Phi\right) + \frac{\sqrt{3}\cos\theta}{r\sqrt{\chi(r)}\sin\theta}\hat{\gamma}^2\left(jr^2 + 2jq - \frac{aq}{2r^2}\right) + \left[\frac{2g(r)}{r\sqrt{h(r)V(r)}}\hat{\gamma}^0 - \frac{2\cot\theta}{r\sqrt{k(r)}}\hat{\gamma}^2 + \frac{1}{\sqrt{h(r)}}\hat{\gamma}^4\right] \times \frac{\sqrt{3}}{2}\left(jr^2 + 2jq - \frac{aq}{2r^2}\right)\right)\Psi + \frac{m}{\hbar}\Psi = 0. \tag{65}$$

Here we also omitted the spin connection terms.

Now we suppose that the wave function takes the same form as in the previous case: (52). Having substituted the wave function (52) into Eq. (65) we arrive at

$$B\left[-\frac{2\sqrt{h(r)}}{r\sqrt{V(r)}}\partial_t I_\uparrow - \frac{\sqrt{V(r)}}{\chi(r)}\partial_r I_\uparrow - \frac{2g(r)}{r\sqrt{h(r)V(r)}}\partial_\psi I_\uparrow + \frac{2e\sqrt{h(r)}}{r\sqrt{V(r)}}\left(\frac{\sqrt{3}q}{2r^2} - \Phi\right) + \frac{2eg(r)}{r\sqrt{h(r)V(r)}}\frac{\sqrt{3}}{2}\left(jr^2 + 2jq - \frac{aq}{2r^2}\right)\right]$$
$$+ A\left[\frac{1}{\sqrt{h(r)}}\partial_\psi I_\uparrow - \frac{e}{\sqrt{h(r)}}\frac{\sqrt{3}}{2}\left(jr^2+2jq-\frac{aq}{2r^2}\right)+\tilde{m}\right] = 0, \tag{66}$$

$$\frac{2B}{r\sqrt{\chi(r)}}\left[-\partial_\theta I_\uparrow - \frac{i}{\sin\theta}\partial_\varphi I_\uparrow - i\cot\theta\partial_\psi I_\uparrow\right] = 0, \tag{67}$$

$$A\left[\frac{2\sqrt{h(r)}}{r\sqrt{V(r)}}\partial_t I_\uparrow - \frac{\sqrt{V(r)}}{\chi(r)}\partial_r I_\uparrow + \frac{2g(r)}{r\sqrt{h(r)V(r)}}\partial_\psi I_\uparrow - \frac{2e\sqrt{h(r)}}{r\sqrt{V(r)}}\left(\frac{\sqrt{3}q}{2r^2} - \Phi\right) - \frac{\sqrt{3}eg(r)}{r\sqrt{h(r)V(r)}}\left(jr^2+2jq-\frac{aq}{2r^2}\right)\right]$$
$$+ B\left[-\frac{1}{\sqrt{h(r)}}\partial_\psi I_\uparrow + \frac{e}{\sqrt{h(r)}}\frac{\sqrt{3}}{2}\left(jr^2+2jq-\frac{aq}{2r^2}\right)+\tilde{m}\right]=0, \tag{68}$$

$$\frac{2A}{r\sqrt{\chi(r)}}\left[-\partial_\theta I_\uparrow - \frac{i}{\sin\theta}\partial_\varphi I_\uparrow - i\cot\theta\partial_\psi I_\uparrow\right] = 0. \tag{69}$$

Equations (67) and (69) show that angular variables can be separated from the radial one. So the action takes the form (57) again. Then the pair of Eqs. (66) and (68) give rise to the following system:

$$B\left[\frac{2\sqrt{h(r)}}{r\sqrt{V(r)}}E - \frac{\sqrt{V(r)}}{\chi(r)}R'(r) - \frac{2g(r)}{r\sqrt{h(r)V(r)}}L + \frac{2e\sqrt{h(r)}}{r\sqrt{V(r)}}\left(\frac{\sqrt{3}q}{2r^2} - \Phi\right) + \frac{2eg(r)}{r\sqrt{h(r)V(r)}}\frac{\sqrt{3}}{2}\left(jr^2 + 2jq - \frac{aq}{2r^2}\right)\right]$$
$$+A\left[\frac{1}{\sqrt{h(r)}}\left(L-\frac{\sqrt{3}}{2}e\left(jr^2+2jq-\frac{aq}{2r^2}\right)\right)+\tilde{m}\right] = 0, \tag{70}$$

$$A\left[-\frac{2\sqrt{h(r)}}{r\sqrt{V(r)}}E - \frac{\sqrt{V(r)}}{\chi(r)}R'(r) + \frac{2g(r)}{r\sqrt{h(r)V(r)}}L - \frac{2e\sqrt{h(r)}}{r\sqrt{V(r)}}\left(\frac{\sqrt{3}q}{2r^2} - \Phi\right) - \frac{\sqrt{3}eg(r)}{r\sqrt{h(r)V(r)}}\left(jr^2 + 2jq - \frac{aq}{2r^2}\right)\right]$$
$$+B\left[-\frac{1}{\sqrt{h(r)}}\left(L-\frac{\sqrt{3}}{2}e\left(jr^2+2jq-\frac{aq}{2r^2}\right)\right)+\tilde{m}\right] = 0. \tag{71}$$

Equations (67) and (69) are identical and lead to an equation for the angular part of the action. This equation is the same as for the previous case, (59) and (61):

$$\Theta'(\theta) + \frac{iJ}{\sin\theta} - iL\cot\theta = 0. \tag{72}$$

So the angular part of the action takes the same form as for the metric (4).

In order to get an equation for the derivative of the radial part of the action we make use of the same arguments as in previous case. So we arrive at the equation

$$R'(r) = \frac{2\chi(r)\sqrt{h(r)}}{rV(r)}\left[\left(E + e\left(\frac{\sqrt{3}q}{2r^2} - \Phi\right) - \frac{g(r)}{h(r)}\left[L - \frac{e\sqrt{3}}{2}\left(jr^2+2jq-\frac{aq}{2r^2}\right)\right]\right)^2 + \frac{r^2V(r)}{4h(r)} \times \left(\tilde{m}^2 - \frac{1}{h(r)}\left[L+\frac{\sqrt{3}}{2}e\left(jr^2+2jq-\frac{aq}{2r^2}\right)\right]^2\right)\right]^{1/2}. \quad (73)$$

Having integrated the latter equation in the vicinity of an horizon point and taking the imaginary part we obtain Eq. (43). So we conclude that the temperature for tunnelling Dirac fermions in the case of a Gödel universe will be the same as for scalar particles and will take form (45).

4 Conclusions

We have investigated tunnelling of scalar particles and Dirac fermions from the squashed charged rotating black holes in the five-dimensional case (4). The same procedure has been accomplished for a similar type of black holes, but in a Gödel universe (32). To consider tunnelling of the scalar particles we have made use of the Hamilton–Jacobi approach which is based on the examination of the Hamilton–Jacobi Eq. (3). As we noted earlier the Hamilton–Jacobi equation we used is the quasi-classical limit of the Klein–Gordon Eq. (1), which describes scalar particles in quantum mechanics. To find the temperature of a black hole the imaginary part of the action which satisfies the Hamilton–Jacobi equation should be found. The fact that the action takes complex values is a direct consequence of the tunnelling process through the horizon. From the point of view of mathematics, complex values for the action appear due to the integration on the interval which includes a pole of integrand (the horizon point is a simple pole for the integrand). The imaginary part of the action allows us to obtain the Boltzmann factor when we calculate the tunnelling probability (29). The expressions for the temperature of the black hole that we have obtained here take the same form as was obtained by another method [35,36].

To consider tunnelling of Dirac particles the approach proposed by Kerner and Mann [11] has been employed. In the case of fermions we also restrict ourselves to the quasi-classical approximation. For this purpose we have chosen the wave function of the Dirac equation in the form (52) and supposed that coefficients $A(t,r,\theta,\phi,\psi)$ and $B(t,r,\theta,\phi,\psi)$ are constant because we take into consideration only the terms proportional to $\hbar^{-1}$. From the written system of equations it follows immediately that the variables can be separated. Then similarly to the case of scalar particles we have singled out the equation for the derivative of the radial part of the action $I_{\uparrow}$. We note that the obtained expressions for the radial part of the action in the vicinity of horizon for Dirac fermions are identical to the corresponding expressions for scalar particles. It should be noticed that the angular part of the action which corresponds to the angle θ can take complex values. But in comparison to the radial part where the imaginary parts for outgoing and ingoing particles take opposite signs, for the angular part $\Theta(\theta)$ they take the same signs and can be cancelled out. So the angular part of the action does not influence the determination of the temperature. As a consequence, the temperature we have found for tunnelling fermions is the same as for scalar particles.

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